Solution to the non-homogenous PDE from class:

$\boldsymbol{xu_x + yu_y = 1 + y^2}$ with boundary condition $\boldsymbol{u(x,1) = x + 1 = f(x)}$
(I add the $= f(x)$ just to point out that the initial condition depends only on x, while our final solution will depend on both x and y).
We start by forming the characteristic equations:

$\frac{dx}{ds} = x$ (you can use dt instead of ds if you want)
$\frac{dy}{ds} = y$
$\frac{du}{ds} = 1 + y^2$

Integrating each of these in turn, we get:
$\frac{dx}{ds} = x \implies ln(x) = s + c \implies x = c_1e^s$
$\frac{dy}{dx} = y \implies ln(y) = s + c \implies y = c_2e^s$

For $\frac{du}{ds} = 1 + y^2$, note that s and y are NOT independent so you cannot treat y as a constant and integrate by ds. Instead, note that $\frac{dy}{ds} = y \implies ds = \frac{1}{y}dy$.
So then:
$\frac{du}{ds} = 1 + y^2 \implies du = 1 + y^2\ ds = \frac{1+y^2}{y}dy = \frac{1}{y} + y\ dy = ln(y) + \frac{y^2}{2} + c_3.$

Now we apply the initial conditions to solve for c_1, c_2 and c_3.

We have $y(s = 0) = 1$ from the boundary condition. So:

$$c_2e^0 = 1 \implies c_2 = 1.$$

For $x(s = 0)$, you might think that the boundary condition gives $x(s = 0) = x + 1$, but this is not quite true as then you would write $x = x$ which is quite useless. Instead, x (which is a function) has one fixed value at $s = 0$, which we can call x_0 (or as Alec called it in class, B) so that:

$$x(s = 0) = x_0 = c_1e^0 \implies c_1 = x_0.$$

Similarly, $u(s = 0) = ln(y) + y^2 + c_3 = 1 + x(s = 0) = 1 + x_0$ (NOT $1 + x$!)

$$\implies c_3 = 1 + x_0 - \frac{1}{2} = \frac{1}{2} + x_0.$$

Now we substitute in c_1, c_2, c_3 to get:
$x = x_0e^s$
$y = e^s$
$u = ln(y) + \frac{y^2}{2} + \frac{1}{2} + x_0$
.
We have three equations with two unknowns (s and x_0) so we can eliminate them to get an equation of u in terms of x and y only - this is the solution!
So $u = ln(y) + \frac{y^2}{2} + \frac{1}{2} + x_0$

$= ln(y) + \frac{y^2}{2} + \frac{1}{2} + xe^{-s}$ (substituting $x = x_0 e^s$)
$= ln(y) + \frac{y^2}{2} + \frac{1}{2} + \frac{x}{y}$ (substituting $y = e^s$)

so we have our final solution $u(x, y) = ln(y) + \frac{y^2}{2} + \frac{1}{2} + \frac{x}{y}$.

It's easy to check if the solution is right by substituting it into the original equation:
$xu_x + yu_y = 1 + y^2$
with $u_x = \frac{1}{y}$ and $u_y = \frac{1}{y} + y + \frac{-x}{y^2}$
and so $xu_x + yu_y = \frac{x}{y} + \frac{y}{y} + y^2 - \frac{xy}{y^2} = 1 + y^2$, as required.

To recap - the steps to solve a PDE:
1) Form the characteristic equations dx/ds, dy/ds, du/ds etc noting that x,y,u may not be independent of s for integration purposes.
2) Each of the char. equations should have solutions that depend on general constants c_1, c_2 etc - solve for these.
3) Put in the initial condition to generate x_0 - you will now have two unknowns (s and x_0).
4) Solve for u, which will likely be in terms of (x, y, x_0, s), and then substitute until you can get rid of s and x_0 and then you will only have u in terms of (x, y) - this is your solution.