Complex Numbers and Coordinate transformations
WHOI Math Review

Isabela Le Bras

July 27, 2015

Class outline:

1. Complex number algebra
2. Complex number application
3. Rotation of coordinate systems
4. Polar and spherical coordinates

1 Complex number algebra

Complex numbers are a combination of real and imaginary numbers. Imaginary numbers are based around the definition of i, $i = \sqrt{-1}$. They are useful for solving differential equations; they carry twice as much information as a real number and there exists a useful framework for handling them.

To add and subtract complex numbers, group together the real and imaginary parts.

For example, $(4+3i)+(3+2i) = 7+5i$.

Try a few examples:

- (9 + 3i) - (4 + 7i) =
- (6i) + (8 + 2i) =
- (7 + 7i) - (9 - 9i) =

To multiply, multiply all components by each other, and use the fact that $i^2 = -1$ to simplify.

For example, $(3-2i)(4+3i) = 12+9i-8i-6i^2 = 18+i$

To divide, multiply the numerator by the complex conjugate of the denominator. The complex conjugate is formed by multiplying the imaginary part of the complex number by -1, and is often denoted by a star, i.e. $(6+3i)^* = (6-3i)$. The reason this works is easier to see in the complex plane.

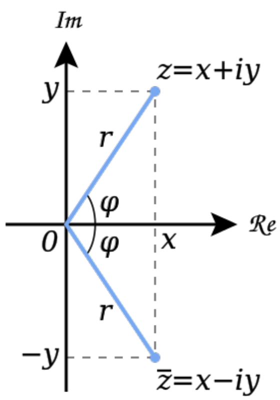
Here is an example of division: $(4+2i)/(3-i) = (4+2i)(3+i) = 12+4i+6i+2i^2 = 10+10i$

Try a few examples of multiplication and division:

- (4 + 7i)(2 + i) =

- (5 + 3i)(5 - 2i) =
- (6 -2i)/(4 + 3i) =
- (3 + 2i)/(6i) =

We often think of complex numbers as living on the plane of real and imaginary numbers, and can write them as $re^{i\theta} = r(cos(\theta) + isin(\theta))$. One of the axes is the imaginary component and the other is the real component:



This also simplifies their multiplication and allows a visual representation of it. To multiply complex numbers on the complex plane, you sum their arguments (θ) and multiply their magnitudes (r). Their addition and subtraction functions as with vector addition and subtraction, and is easier to do in cartesian coordinates as before.

Try converting these complex numbers into another coordinate system:

- $5+5i$
- $200i$
- $6-7i$
- $4(cos(\pi/3)+isin(\pi/3))$
- $5e^{i\pi/4}$

Here is an example of multiplication in polar coordinates/ complex space: $e^{i\pi/2} * 6e^{-i\pi/4} = 6e^{i\pi/4}$

Try a few examples:

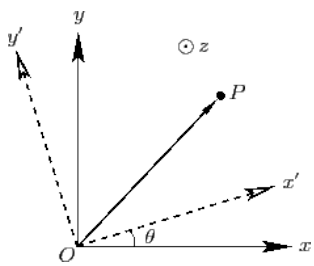
- $2e^{i\pi/4} * 4e^{i\pi/4} =$
- $4e^{i\pi/5}/3e^{i5\pi/6} =$
- $3(cos(\pi/2)+isin(\pi/2)) * 2(cos(\pi/4)+isin(\pi/4)) =$

2 Complex number application

The Ekman Spiral, from https : //www.rsmas.miami.edu/personal/lbeal/MPO%20503/Lecture%2011.xhtml

3 Rotation of coordinate systems

Rotating coordinate systems is very useful, when analyzing data along a measurement line or some axis that makes more sense than north and east because of topography, the point of view of a whale, or some other natural phenomenon. To change into a coordinate system that is offset by some angle θ, you can use the following transformation. This is just like multiplication by a complex number of magnitude one.



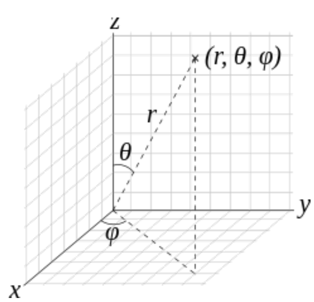
$$a'_x = a_x cos(\theta) + a_y sin(\theta)$$
$$a'_y = -a_y sin(\theta) + a_y cos(\theta)$$

The rotation matrix can also be written out:

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

4 Spherical coordinates

Spherical coordinates are the three dimensional extension of the polar coordinates we have already been using. They provide an alternate description of a point in the 3d plane, which can be useful for doing integrals over spherical or conical shapes and often make more sense for doing math on the spherical earth.



$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\phi = tan^{-1}(y/x)$$
$$\theta = cos^{-1}(z/r)$$

Below is a pretty complex and PO specific example of coordinates used for Geophysical Fluid Dynamics. Feel free to work through it and/or keep as a reference.

To summarize, the starting point is the following COMPLETE SET OF EQUATIONS:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{1}$$

$$\frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{F}_b \tag{2}$$

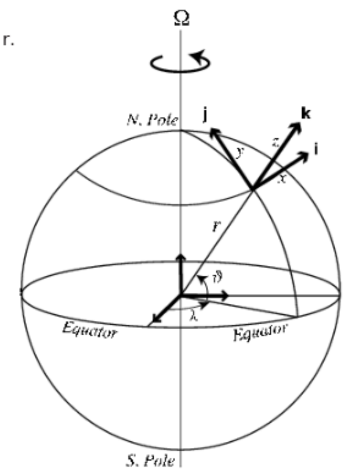
$$\rho = \rho(p, T, \tau) \tag{3}$$

$$\frac{\partial (\rho \tau)}{\partial t} + \nabla \cdot (\rho \tau \mathbf{u}) = \rho S^{(\tau)} \tag{4}$$

$$\frac{D\rho}{Dt} - \frac{1}{c_s^2}\frac{Dp}{Dt} = Q(\rho) \tag{5}$$

(a) Momentum equation is for velocity vector, hence, it can be written as 3 equations for velocity components (scalars).

(b) We ended up with 7 equations and 7 unknowns: $u, v, w, p, \rho, T, \tau$.



• **Spherical coordinates** are natural for GFD: longitude, λ, latitude, θ, and altitude, r.

Material derivative for a scalar quantity φ in spherical coordinates is:

$$\frac{D}{Dt} = \frac{\partial \varphi}{\partial t} + \frac{u}{r\cos\theta}\frac{\partial \varphi}{\partial \lambda} + \frac{v}{r}\frac{\partial \varphi}{\partial \theta} + w\frac{\partial \varphi}{\partial r},$$

where the velocity in terms of the unit vectors is:

$$\mathbf{u} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w, \qquad (u, v, w) \equiv \left(r\cos\theta\frac{D\lambda}{Dt}, r\frac{D\theta}{Dt}, \frac{Dr}{Dt}\right)$$

Vector analysis provides differential operators in spherical coordinates acting on scalar, φ, or vector, $\mathbf{B} = \mathbf{i}B^{\lambda} + \mathbf{j}B^{\theta} + \mathbf{k}B^{r}$, fields:

$$\nabla \cdot \mathbf{B} = \frac{1}{\cos\theta}\left(\frac{1}{r}\frac{\partial B^{\lambda}}{\partial \lambda} + \frac{1}{r}\frac{\partial (B^{\theta}\cos\theta)}{\partial \theta} + \frac{\cos\theta}{r^2}\frac{\partial (r^2 B^{r})}{\partial r}\right),$$

$$\nabla \varphi = \mathbf{i}\frac{1}{r^2\cos\theta}\frac{\partial \varphi}{\partial \lambda} + \mathbf{j}\frac{1}{r}\frac{\partial \varphi}{\partial \theta} + \mathbf{k}\frac{\partial \varphi}{\partial r},$$

$$\nabla^2 \varphi \equiv \nabla \cdot \nabla \varphi = \frac{1}{r^2\cos\theta}\left[\frac{1}{\cos\theta}\frac{\partial^2 \varphi}{\partial \lambda^2} + \frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial \varphi}{\partial \theta}\right) + \cos\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial \varphi}{\partial r}\right)\right],$$

$$\nabla \times \mathbf{B} = \frac{1}{r^2\cos\theta}\begin{vmatrix} \mathbf{i}\,r\cos\theta & \mathbf{j}\,r & \mathbf{k} \\ \partial/\partial\lambda & \partial/\partial\theta & \partial/\partial r \\ B^{\lambda} r\cos\theta & B^{\theta} r & B^{r} \end{vmatrix}.$$

$$\nabla^2 \mathbf{B} = \nabla(\nabla \cdot \mathbf{B}) - \nabla \times (\nabla \times \mathbf{B}).$$

In spherical coordinates, the directions of the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ change with location (this doesn't happen in Cartesian coordinates!), therefore, their material derivatives are not zero:

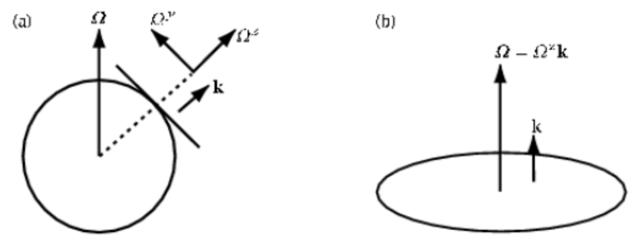
$$\frac{D\mathbf{u}}{Dt} = \frac{Du}{Dt}\mathbf{i} + \frac{Dv}{Dt}\mathbf{j} + \frac{Dw}{Dt}\mathbf{k} + u\frac{D\mathbf{i}}{Dt} + v\frac{D\mathbf{j}}{Dt} + w\frac{D\mathbf{k}}{Dt} = \frac{Du}{Dt}\mathbf{i} + \frac{Dv}{Dt}\mathbf{j} + \frac{Dw}{Dt}\mathbf{k} + \boldsymbol{\Omega}_{flow} \times \mathbf{u}, \qquad (*)$$

where $\boldsymbol{\Omega}_{flow}$ is rotation rate (relative to the centre of Earth) of a unit vector moving with the fluid flow:

$$\frac{D\mathbf{i}}{Dt} = \boldsymbol{\Omega}_{flow} \times \mathbf{i},$$

$$\frac{D\mathbf{j}}{Dt} = \boldsymbol{\Omega}_{flow} \times \mathbf{j},$$

$$\frac{D\mathbf{k}}{Dt} = \boldsymbol{\Omega}_{flow} \times \mathbf{k}.$$



Let's find $\boldsymbol{\Omega}_{flow}$ by noting that a rotation around the Earth's rotation axis can be written as (see Figure): $\boldsymbol{\Omega}_{||} = \Omega_{||}(\mathbf{j}\cos\theta + \mathbf{k}\sin\theta)$.

This rotation rate comes from a zonally moving fluid element:

$$u\delta t = r\cos\theta\delta\lambda \quad \rightarrow \quad \Omega_{||} \equiv \frac{\delta\lambda}{\delta t} = \frac{u}{r\cos\theta} \quad \Longrightarrow \quad \boldsymbol{\Omega}_{||} = \frac{u}{r\cos\theta}(\mathbf{j}\cos\theta + \mathbf{k}\sin\theta) = \mathbf{j}\frac{u}{r} + \mathbf{k}\frac{u\tan\theta}{r}.$$

Rotation rate in the perpendicular direction is

$$\boldsymbol{\Omega}_{\perp} = -\mathbf{i}\frac{v}{r} \quad \Longrightarrow \quad \boldsymbol{\Omega}_{flow} = -\mathbf{i}\frac{v}{r} + \mathbf{j}\frac{u}{r} + \mathbf{k}\frac{u\tan\theta}{r} \quad \Longrightarrow$$

$$\frac{D\mathbf{i}}{Dt} = \boldsymbol{\Omega}_{flow} \times \mathbf{i} = \frac{u}{r\cos\theta}(\mathbf{j}\sin\theta - \mathbf{k}\cos\theta), \qquad \frac{D\mathbf{j}}{Dt} = -\mathbf{i}\frac{u}{r}\tan\theta - \mathbf{k}\frac{v}{r}, \qquad \frac{D\mathbf{k}}{Dt} = \mathbf{i}\frac{u}{r} + \mathbf{j}\frac{v}{r}$$

$$(*) \quad \Longrightarrow \quad \frac{D\mathbf{u}}{Dt} = \mathbf{i}\left(\frac{Du}{Dt} - \frac{uv\tan\theta}{r} + \frac{uw}{r}\right) + \mathbf{j}\left(\frac{Dv}{Dt} - \frac{u^2\tan\theta}{r} + \frac{vw}{r}\right) + \mathbf{k}\left(\frac{Dw}{Dt} - \frac{u^2+v^2}{r}\right)$$

The additional quadratic terms are called *metric terms*. The last effort is to write the Coriolis force in terms of the unit vectors:

$$2\boldsymbol{\Omega} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2\Omega\cos\theta & 2\Omega\sin\theta \\ u & v & w \end{vmatrix} = \mathbf{i}(2\Omega w\cos\theta - 2\Omega v\sin\theta) + \mathbf{j}\,2\Omega u\sin\theta - \mathbf{k}\,2\Omega u\cos\theta.$$

By combining the derived terms, we obtain the governing equations (with gravity g):

$$
\begin{aligned}
\frac{Du}{Dt} - \ 2\Omega + \frac{u}{r\cos\theta} \ (v\sin\theta - w\cos\theta) &= -\frac{1}{\rho r\cos\theta}\frac{\partial p}{\partial \lambda}, \\
\frac{Dv}{Dt} + \frac{wv}{r} + \ 2\Omega + \frac{u}{r\cos\theta} \ u\sin\theta &= -\frac{1}{\rho r}\frac{\partial p}{\partial \theta}, \\
\frac{Dw}{Dt} - \frac{u^2 + v^2}{r} - 2\Omega u\cos\theta &= -\frac{1}{\rho}\frac{\partial p}{\partial r} - g, \\
\frac{\partial \rho}{\partial t} + \frac{1}{r\cos\theta}\frac{\partial(u\rho)}{\partial \lambda} + \frac{1}{r\cos\theta}\frac{\partial(v\rho\cos\theta)}{\partial \theta} + \frac{1}{r^2}\frac{\partial(r^2 w\rho)}{\partial r} &= 0.
\end{aligned}
$$

(a) Metric terms are relatively small on the surface of a large planet $(r \to R_0)$ and can be neglected for many process studies;

(b) Terms with w can be neglected, if common hydrostatic approximation is made (see later).

• **Local Cartesian approximation**. For mathematical simplicity and process studies, the equations can be written for a plane tangent to the planetary surface. Then, the momentum equations with $\Omega = (\Omega_x, \Omega_y, \Omega_z)$ can be written as

$$
\frac{Du}{Dt} + 2(\Omega_y w - \Omega_z v) = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \qquad \frac{Dv}{Dt} + 2(\Omega_z u - \Omega_x w) = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \qquad \frac{Dw}{Dt} + 2(\Omega_x v - \Omega_y u) = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g,
$$

where $\Omega_x \equiv 0$. Next, neglect Ω_y, because its effect (upward/downward deflection of fluid particles) is small and introduce *Coriolis parameter*: $f \equiv 2\Omega_z = 2\Omega\sin\theta$.

The resulting momentum equations are: $\boxed{\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \quad \frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \quad \frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g}$

(a) Theoreticians often use f -*plane approximation*: $f = f_0$ (constant).

(b) Planetary sphericity is often accounted for by β-*plane approximation*: $f(y) = f_0 + \beta y$.

(c) *Boundary and initial conditions*: The governing equations (or their approximations) are to be solved subject to those.

Source: Pavel Berloff, http : //wwwf.imperial.ac.uk/ pberloff/gfd_lectures.pdf