Complex Numbers and Coordinate transformations
WHOI Math Review

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July 27, 2015

Class outline:
1. Complex number algebra
2. Complex number application
3. Rotation of coordinate systems
4. Polar and spherical coordinates

1 Complex number algebra

Complex numbers are a combination of real and imaginary numbers. Imaginary numbers are based around the definition of $i$, $i = \sqrt{-1}$. They are useful for solving differential equations; they carry twice as much information as a real number and there exists a useful framework for handling them.

To add and subtract complex numbers, group together the real and imaginary parts.

For example, $(4 + 3i) + (3 + 2i) = 7 + 5i$.

Try a few examples:

- $(9 + 3i) - (4 + 7i) =$
- $(6i) + (8 + 2i) =$
- $(7 + 7i) - (9 - 9i) =$

To multiply, multiply all components by each other, and use the fact that $i^2 = -1$ to simplify.

For example, $(3 - 2i)(4 + 3i) = 12 + 9i - 8i - 6i^2 = 18 + i$

To divide, multiply the numerator by the complex conjugate of the denominator. The complex conjugate is formed by multiplying the imaginary part of the complex number by -1, and is often denoted by a star, i.e. $(6 + 3i)^* = (6 - 3i)$. The reason this works is easier to see in the complex plane.

Here is an example of division: $(4 + 2i)/(3 - i) = (4 + 2i)(3 + i) = 12 + 4i + 6i + 2i^2 = 10 + 10i$

Try a few examples of multiplication and division:

- $(4 + 7i)(2 + i) =$
• $(5 + 3i)(5 - 2i)$ =
• $(6 -2i)/(4 + 3i)$ =
• $(3 + 2i)/(6i)$ =

We often think of complex numbers as living on the plane of real and imaginary numbers, and can write them as $re^{\theta} = r(cos(\theta) + isin(\theta))$. One of the axes is the imaginary component and the other is the real component:

This also simplifies their multiplication and allows a visual representation of it. To multiply complex numbers on the complex plane, you sum their arguments ($\theta$) and multiply their magnitudes ($r$). Their addition and subtraction functions as with vector addition and subtraction, and is easier to do in cartesian coordinates as before.

Try converting these complex numbers into another coordinate system:

• $5 + 5i$
• $200i$
• $6 - 7i$
• $4(cos(\pi/3) + isin(\pi/3))$
• $5e^{i\pi/4}$

Here is an example of multiplication in polar coordinates/ complex space: $e^{i\pi/2} \times 6e^{-i\pi/4} = 6e^{i\pi/4}$

Try a few examples:

• $2e^{i\pi/4} \times 4e^{i\pi/4} =$
• $4e^{i\pi/5} / 3e^{i\pi/6} =$
• $3(cos(\pi/2) + isin(\pi/2)) \times 2(cos(\pi/4) + isin(\pi/4)) =$

2 Complex number application

The Ekman Spiral, from https : //www.rsmas.miami.edu/personal/lbeal/MPO%20503/Lecture%2011.xhtml
3 Rotation of coordinate systems

Rotating coordinate systems is very useful, when analyzing data along a measurement line or some axis that makes more sense than north and east because of topography, the point of view of a whale, or some other natural phenomenon. To change into a coordinate system that is offset by some angle $\theta$, you can use the following transformation. This is just like multiplication by a complex number of magnitude one.

$$a_x' = a_x \cos(\theta) + a_y \sin(\theta)$$
$$a_y' = -a_y \sin(\theta) + a_x \cos(\theta)$$

The rotation matrix can also be written out:

$$R = \begin{bmatrix}
\cos\theta & -\sin\theta \\
\sin\theta & \cos\theta 
\end{bmatrix}$$

4 Spherical coordinates

Spherical coordinates are the three dimensional extension of the polar coordinates we have already been using. They provide an alternate description of a point in the 3d plane, which can be useful for doing integrals over spherical or conical shapes and often make more sense for doing math on the spherical earth.

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\phi = \tan^{-1}(y/x)$$
$$\theta = \cos^{-1}(z/r)$$

Below is a pretty complex and PO specific example of coordinates used for Geophysical Fluid Dynamics. Feel free to work through it and/or keep as a reference.
To summarize, the starting point is the following COMPLETE SET OF EQUATIONS:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) &= 0, \\
\frac{\partial \mathbf{u}}{\partial t} + 2\Omega \times \mathbf{u} &= -\frac{1}{\rho} \nabla \rho + \mathbf{v} \nabla^2 \mathbf{u} + \mathbf{F}_b, \\
\rho &= \rho(\mathbf{R}, T, \tau), \\
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) &= 0, \\
\frac{\partial \mathbf{p}}{\partial t} - \frac{1}{ho} \nabla \cdot \mathbf{p} &= 0.
\end{align*}
\]

(a) Momentum equation is for velocity vector, hence, it can be written as 3 equations for velocity components (scalars).

(b) We ended up with 7 equations and 7 unknowns: \( u, v, w, p, \rho, T, \tau \).

**Spherical coordinates** are natural for GFD: longitude, \( \lambda \), latitude, \( \beta \), and altitude, \( t \).

Material derivative for a scalar quantity \( \phi \) in spherical coordinates is:

\[
\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \frac{\mathbf{u} \cdot \nabla \phi}{r \cos \theta} + \frac{\partial}{\partial r} \left( \frac{\partial \phi}{\partial r} \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{r \sin^2 \theta} \frac{\partial \phi}{\partial \tau},
\]

where the velocity in terms of the unit vectors:

\( \mathbf{u} = u \hat{\mathbf{r}} + v \hat{\mathbf{\theta}} + w \hat{\mathbf{\tau}} \), \( (u, v, w) \equiv \frac{\partial \mathbf{R}}{\partial t} \frac{\partial \beta}{\partial \beta} + \frac{\partial \mathbf{R}}{\partial \tau} \frac{\partial \tau}{\partial t} \)

Vector analysis provides differential operators in spherical coordinates acting on scalar, \( \phi \), or vector, \( \mathbf{B} \):

\( \nabla \cdot \mathbf{B} = \frac{1}{r \cos \theta} \frac{\partial}{\partial r} \left( r \cos \theta \frac{\partial \phi}{\partial r} \right) + \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \theta} + \frac{1}{r \sin^2 \theta} \frac{\partial \phi}{\partial \tau} \),

\( \nabla \times \mathbf{B} = \frac{1}{r \cos \theta} \left( \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \phi}{\partial \theta} \right) - \frac{\partial \phi}{\partial r} \right) - \frac{1}{r} \frac{\partial \phi}{\partial \tau} \),

\( \nabla^2 \phi = \nabla (\nabla \cdot \mathbf{B}) - \nabla \times (\nabla \times \mathbf{B}) \).
In spherical coordinates, the directions of the unit vectors \( i, j, k \) change with location (this doesn't happen in Cartesian coordinates!), therefore, their material derivatives are not zero:

\[
\frac{\partial u}{\partial t} \frac{\partial}{\partial t} + \frac{\partial u}{\partial r} \frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial u}{\partial \theta} \frac{\partial}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial u}{\partial \phi} \frac{\partial}{\partial \phi} = \frac{\partial v}{\partial t} \frac{\partial}{\partial t} + \frac{\partial v}{\partial r} \frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \theta} \frac{\partial}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial v}{\partial \phi} \frac{\partial}{\partial \phi} + \frac{\partial w}{\partial t} \frac{\partial}{\partial t} + \frac{\partial w}{\partial r} \frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial w}{\partial \theta} \frac{\partial}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial w}{\partial \phi} \frac{\partial}{\partial \phi} + \Omega_{\text{rot}} \times \mathbf{u}.
\]

(4)

where \( \Omega_{\text{rot}} \) is rotation rate relative to the centre of Earth of a unit vector moving with the fluid flow:

\[
\frac{\partial \mathbf{u}}{\partial t} = \Omega_{\text{rot}} \times \mathbf{u}.
\]

Let's find \( \Omega_{\text{rot}} \) by noting that if rotation around the Earth's rotation axis can be written as (see Figure), \( \Omega_1 = \Omega (j \cos \theta + k \sin \theta) \).

This rotation rate comes from a vertically moving fluid element:

\[
\mathbf{u} \times \mathbf{x} = \mathbf{r} \cos \Theta \rightarrow \mathbf{\Omega} \times \mathbf{r} = \frac{\partial \mathbf{r}}{\partial \Theta} = \frac{\mathbf{u}}{r \cos \Theta} \rightarrow \mathbf{\Omega} = \frac{\mathbf{u}}{r \cos \Theta} (j \cos \theta + k \sin \theta) = \frac{\mathbf{u}}{r} + \frac{\mathbf{u} \tan \theta}{r}.
\]

Rotation rate in the perpendicular direction is

\[
\Omega_1 = -\frac{\mathbf{u}}{r} \rightarrow \Omega_{\text{rot}} = -\frac{\mathbf{u}}{r} + \frac{\mathbf{u} \tan \theta}{r}
\]

The additional quadratic terms are called metric terms. The last effect is to write the Coriolis force in terms of the unit vectors:

\[
2\Omega \times \mathbf{u} = 0 \rightarrow 2\Omega \cos \Theta 2\Omega \sin \Theta = \mathbf{i} (2\Omega w \cos \Theta - 2\Omega v \sin \Theta) + \mathbf{j} 2\Omega u \sin \Theta - k 2\Omega u \cos \Theta.
\]

\[
\mathbf{u} \times \mathbf{v} = \mathbf{w}.
\]
By combining the derived terms, we obtain the governing equations (with gravity $g$):

$$
\frac{Du}{Dt} - 2\Omega + \frac{u}{r \cos \theta} (v \sin \theta - w \cos \theta) = -\frac{1}{\rho r \cos \theta} \frac{\partial p}{\partial \lambda}, \\
\frac{Dv}{Dt} + \frac{w}{r} + 2\Omega + \frac{u}{r \cos \theta} u \sin \theta = -\frac{1}{\rho \theta} \frac{\partial p}{\partial \theta}, \\
\frac{ Dw}{Dt} - \frac{u^2 + v^2}{r} - 2\Omega u \cos \theta = -\frac{1}{\rho r} \frac{\partial p}{\partial r} - g, \\
\frac{\partial \rho}{\partial t} + \frac{1}{r \cos \theta} \frac{\partial (\rho u)}{\partial \lambda} + \frac{1}{r \cos \theta} \frac{\partial (\rho v \cos \theta)}{\partial \theta} + \frac{1}{r^2} \frac{\partial (\rho w)}{\partial r} = 0.
$$

(a) Metric terms are relatively small on the surface of a large planet ($r \rightarrow R_2$) and can be neglected for many process studies.

(b) Terms with $w$ can be neglected, if common hydrostatic approximation is made (see later).

• **Local Cartesian approximation.** For mathematical simplicity and process studies, the equations can be written for a plane tangent to the planetary surface. Then, the momentum equations with $\Omega = (\Omega_x, \Omega_y, \Omega_z)$ can be written as

$$
\frac{Du}{Dt} + 2(\Omega_y w - \Omega_z v) = -\frac{1}{\rho \theta} \frac{\partial p}{\partial \lambda}, \\
\frac{Dv}{Dt} + 2(\Omega_z u - \Omega_x w) = -\frac{1}{\rho \theta} \frac{\partial p}{\partial \theta}, \\
\frac{ Dw}{Dt} + 2(\Omega_x v - \Omega_y u) = -\frac{1}{\rho \theta} \frac{\partial p}{\partial \theta} - g.
$$

where $\Omega_0 = 0$. Next, neglect $\Omega_z$, because its effect (upward/downward deflection of fluid particles) is small and introduce Coriolis parameter: $f = \Omega_x = 2\Omega \sin \theta$.

The resulting momentum equations are:

$$
\frac{Du}{Dt} - f v = -\frac{1}{\rho \theta} \frac{\partial p}{\partial \lambda}, \\
\frac{Dv}{Dt} + f u = -\frac{1}{\rho \theta} \frac{\partial p}{\partial \theta}, \\
\frac{ Dw}{Dt} = -\frac{1}{\rho \theta} \frac{\partial p}{\partial \theta} - g.
$$

(a) Theoreticians often use $f$-plane approximation: $f = f_0$ (constant).

(b) Planetary sphericity is often accounted for by $\beta$-plane approximation: $f(\phi) = f_0 + \beta \phi$.

(c) **Boundary and initial conditions:** The governing equations (or their approximations) are to be solved subject to those.

Source: Pavel Berloff, http://wwwf.imperial.ac.uk/~pberloff/gfd_lectures.pdf